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Kac–Moody algebras derived from linearisation systems using various reductions and extended to supersymmetry

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Abstract. The hidden symmetries in various integrable models are derived by applying a newly developed method that uses the Riemann–Hilbert transform in various reductions of the linearisation systems. The method is extended to linearisation systems with higher algebras and with supersymmetry.

1. Introduction

Hidden symmetries such as the recently discovered [1] Kac–Moody type, in mostly two-dimensional integrable models, have attracted considerable interest; since they are thought to be responsible for the infinite set of conserved currents in these models. The affine Kac–Moody algebra is defined as

$$[T_a^{(n)}, T_b^{(m)}] = C_{abc} T_C^{(n+m)}$$

where $T_a^{(n)} = T_a \otimes t^n$ $(n \in \mathbb{Z})$ are the generators for the loop group $\mathfrak{G} \otimes C[-t, t]$.

In this paper we shall study a method [2] for deriving these hidden symmetries from the linearisation system of various integrable models. The method will here be extended to systems with more complex algebraic structure and to supersymmetric cases. Such extensions therefore make it plausible that two-dimensional integrable models in general have associated an infinite parameter hidden symmetry.

The method recently developed for finding the hidden symmetries will be reviewed in the next section. It goes briefly as follows: one starts with the linearisation system (i.e. the Lax pair) of the model under investigation. Next one finds a reduction system that together with the Frobenius consistency condition reproduces the equation of motion of the model. The global version of the Kac–Moody generators are the Riemann–Hilbert transformations [3] and once they are guaranteed to satisfy the reduction system they should leave the linearisation system invariant and their infinitesimal limit yields therefore the corresponding Kac–Moody algebra.

In the following section we shall apply this technique to the Dodd–Bullough [4] equation. Originally the technique was applied to the sine-Gordon and the Liouville

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models. Both are two-dimensional, integrable SU(2) systems. By extending the method to the Dodd–Bullough system which possesses an SU(3) symmetry and is a mixture of the two former models it then becomes clear how to extend the techniques to an arbitrary SU(N) symmetry. Furthermore, we shall supersymmetrise the model in order to see how the method works in superspace and with a graded algebra structure. It might even be possible to extend the techniques to higher dimensions such as, for example, the case of the non-relativistic three-dimensional Kadomstov–Petviasvili equation.

2. A method of deriving the hidden symmetries from linearisation systems

In this section we shall review a method [2] proposed by Bohr *et al*, by which hidden symmetries can be derived from the Lax pair of integrable systems. The method was particularly intended for the simple two-dimensional models such as the sine-Gordon and the Liouville model. Before generalising the method to more complicated algebraic structures we shall first introduce the method in the simple case of the sine-Gordon equation.

The strategy is the following. We start with a general Lax pair that can represent a class of two-dimensional integrable models and then find a reduction system that together with the usual Frobenius consistency conditions can reproduce the particular model we want. The reduction system is therefore applied to the linear wavefunction to yield the particular nonlinear equation we want. The Riemann–Hilbert transform gives an analytic continuation of the linear wavefunction in the λ plane (λ is the spectral parameter of the Lax pair). If we therefore apply the reduction system to the Riemann–Hilbert transform this should correspond to our particular model, i.e. the Riemann–Hilbert transform that obeys the reduction system should leave the particular Lax pair invariant. Finally the infinitesimal limit of the Riemann–Hilbert transform gives us the corresponding Kac–Moody algebra as a hidden symmetry.

Let us illustrate this procedure by the sine-Gordon case. We start with a general linearisation system (or Lax pair):

$$\begin{aligned} \partial_x\psi &= U\psi = [U_0 + U_1\lambda]\psi \\ \partial_y\psi &= V\psi = [\lambda^{-1}V_1]\psi. \end{aligned} \tag{1}$$

The Frobenius consistency condition is obtained by differentiating the first equation by y and the last by z and using compatibility $\partial_x\partial_y\psi = \partial_y\partial_x\psi$:

$$\partial_y U_1 = 0, \qquad \partial_y U_0 = [V_1, U_1], \qquad \partial_x V_1 = [U_0, V_1]. \tag{2}$$

The reduction system we now choose is the following

$$\psi \in \mathrm{SL}(2, \mathbb{C}) \tag{3a}$$

$$(\psi(\lambda))^* = \psi(-\bar{\lambda})^{-1} \tag{3b}$$

$$\sigma_3\psi(\lambda)\sigma_3^{-1} = \psi(-\lambda). \tag{3c}$$

(In the next section we shall apply a more general scheme of reduction systems (see [5]), the so-called Z_N, D_N reduction groups.)

By applying the system (3) on the Lax pair in (1), together with (2) we obtain (see [2]) uniquely the specific Lax pair for the sine-Gordon model

$$\begin{aligned}\partial_x\psi &= \frac{1}{2}\left[\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\phi_x + \frac{\lambda}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right]\psi\\ \partial_y\psi &= \frac{1}{2\lambda}\begin{pmatrix}0 & e^{i\phi}\\ e^{-i\phi} & 0\end{pmatrix}\psi.\end{aligned} \tag{4}$$

Applying again the compatibility condition on (4) we obtain the sine-Gordon equation

$$\partial_x\partial_y\phi = \sin\phi.$$

We now apply the Riemann–Hilbert transform to the reduction system (3):

$$\chi_\pm \in \mathrm{SL}(2, \mathbb{C}) \tag{5a}$$

$$[\chi_\pm(\lambda)]^* = -\chi_\pm(-\bar{\lambda}) \tag{5b}$$

$$\sigma_3\chi_\pm(\lambda)\sigma_3^{-1} = \chi_\pm(-\lambda). \tag{5c}$$

The RH-transform $\chi(\lambda)$ is in general given by

$$\begin{aligned}\chi(\lambda) &= 1 - \frac{1}{2\pi i}\int \frac{dt}{t-\lambda}\chi(t)[S(t)S(\lambda)^{-1} - 1]\\ \chi&: \psi \to \psi'.\end{aligned} \tag{6}$$

χ_+ and χ_- denote the analytic transformations inside and outside a contour C (in the complex λ plane). We have introduced S as

$$\chi_+(\lambda) = \chi_-(\lambda)S, \qquad S(\lambda) = \psi(\lambda)\gamma(\lambda)\psi(\lambda)^{-1} \tag{7}$$

where ψ is the wavefunction of λ in (1) and $\gamma(\lambda)$ a general matrix function.

It can now be proven (see [2]) that if the RH-transform and $\gamma(\lambda)$ obey (5) then it leaves the linearisation system invariant; which is rather obvious.

From the constraints in (5) we can now easily derive the basis that spans the subalgebra of the $X(\lambda)$ transformations that leave the sine-Gordon invariant. First we go from the group elements to the algebra which in the infinitesimal limit becomes (θ small)

$$\begin{aligned}\gamma(\lambda) &= \exp[i\theta(\lambda)] \simeq 1 + \theta(\lambda)\\ \theta(\lambda) &\simeq \sum_{-\infty}^{\infty} \lambda^n\theta_n.\end{aligned} \tag{8}$$

If $\gamma(\lambda)$ should obey (5) then we obtain for the Laurent series above:

$$\mathrm{Tr}(\theta(\lambda)) = 0 \qquad \theta_n^* = (-1)^{n+1}\theta_n \tag{9a}$$

$$(\theta(\lambda))^* = -\theta(-\bar{\lambda}) \qquad \sigma_3\theta_n\sigma_3 = (-1)^n\theta_n \tag{9b}$$

$$\sigma_3\theta(\lambda)\sigma_3 = \theta(-\lambda) \tag{9c}$$

and the grading is obtained as powers of λ.

From (9) it follows that the θ transformations can be expanded on the following basis

$$\theta(\lambda) = \sum C_K i\sigma_3\lambda^{2K} + \sum C'_K\sigma_1\lambda^{2K+1} + \sum C''_K\sigma_2\lambda^{2K+1}. \tag{10}$$

It is then clear that the particular Kac–Moody algebra that is infinite parameter for the hidden symmetries of the sine-Gordon model is the following subalgebra:

$$[T^m_a, T^n_b] = e_{abc} T^{m+n}_C$$
$$T^m_C = \begin{cases} \lambda^m \mathrm{i}\sigma_3 \delta_{a,3} & m \text{ even} \\ \lambda^m (\sigma_1 \delta_{a,1} + \sigma_2 \delta_{a,2}) & m \text{ odd} \end{cases} \tag{11}$$

which is a subalgebra of the ordinary Kac-Moody algebra of the chiral model and has a special structure depending on whether the grading is even or odd.

3. Extension to higher order Lax pair

After the above introduction to the procedure of obtaining the hidden symmetries we here proceed to show how the same procedure can be effectively used to deduce the hidden symmetries in the case of nonlinear equations, associated with a $N \times N$ matrix Lax pair with $N > 2$. It has already been observed that it is possible to generate a family of nonlinear equations whose members are the sine-Gordon and Liouville equations. Such a system of equations is known as a generalised Toda system [5]. The equation belonging to this hierarchy next to sine-Gordon is the Dodd-Bullough equation [4] which is known to be associated with a 3×3 Lax pair. In the following we show how our method is useful to extract the structure of the Kac-Moody algebra associated with such a 3×3 system, with the help of a reduction technique. Finally in the next section, we also consider the corresponding grade extension for the supersymmetric case.

Let us consider the Lax pair of the form:

$$\begin{aligned} L_1\psi &= (\partial_x - U_0 - \lambda U_1)\psi = 0 \\ L_2\psi &= (\partial_t - V_0 - \lambda^{-1} V_1)\psi = 0. \end{aligned} \tag{12}$$

The compatibility condition yields

$$\begin{aligned} &U_{0t} - V_{0x} + [U_0, V_0] + [U_1, V_1] = 0 \\ &U_{1t} + [U_1, V_0] = 0, \qquad V_{1x} + [V_1, U_0] = 0. \end{aligned} \tag{13}$$

In (12) and (13) U_0, U_1 and V_0, V_1 are all assumed to be 3×3 matrices.

Our initial problem is to deduce the Dodd-Bullough equation by imposing the reduction condition on the scattering problem (12). The general ansatz as elaborated in [5] reads in our case

$$\begin{aligned} Q^{-1} L_1(\lambda) Q &= L_1(\lambda q) \\ Q^{-1} L_2(\lambda) Q &= -L_2(\lambda q) \end{aligned} \tag{14}$$

and

$$t^{-1}[L^{\mathrm{tr}}_{1,2}(-\lambda)]t = L_{1,2}(\lambda)$$

with

$$Q_{\alpha\beta} = \delta_{\alpha\beta} q^\alpha \qquad q = \exp(2\pi \mathrm{i}/N) \qquad N = 3.$$

Equation (14) yields

$$Q^{-1} U_0 Q = U_0 \qquad Q^{-1} V_0 Q = V_0 \tag{15a}$$

and

$$Q^{-1}U_1Q = qU_1 \qquad Q^{-1}V_1Q = q^{-1}V_1. \tag{15b}$$

The second part of the reduction condition yields

$$\begin{aligned} tU_0t^{-1} &= -U_0^{\mathrm{tr}} & tV_0t^{-1} &= -V_0^{\mathrm{tr}} \\ tU_1t^{-1} &= U_1^{\mathrm{tr}} & tV_1t^{-1} &= V_1^{\mathrm{tr}}. \end{aligned} \tag{16}$$

These equations lead to the following form of the matrices U and V:

$$\begin{aligned} U_0 &= \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & 0 \end{pmatrix} & V_0 &= \begin{pmatrix} a_1 & 0 & 0 \\ 0 & -a_1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ U_1 &= \begin{pmatrix} 0 & \alpha & 0 \\ 0 & 0 & \beta \\ \beta & 0 & 0 \end{pmatrix} & V_1 &= \begin{pmatrix} 0 & 0 & h_1 \\ d_1 & 0 & 0 \\ 0 & h_1 & 0 \end{pmatrix} \end{aligned} \tag{17}$$

with

$$t = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

when used in the compatibility (13) these matrices lead to the equation

$$\theta_{xt} = \mathrm{e}^{-\theta} - \mathrm{e}^{2\theta} \tag{18}$$

if we choose

$$a_1 = \partial\theta/\partial t.$$

Finally the linear problem takes the form:

$$\begin{aligned} \psi_x &= \lambda \cdot \begin{pmatrix} 0 & \mathrm{e}^{2\theta} & 0 \\ 0 & 0 & \mathrm{e}^{-\theta} \\ \mathrm{e}^{-\theta} & 0 & 0 \end{pmatrix}\psi \\ \psi_t &= \left[\begin{pmatrix} \theta_t & 0 & 0 \\ 0 & -\theta_t & 0 \\ 0 & 0 & 0 \end{pmatrix} + \lambda^{-1}\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\right]. \end{aligned} \tag{19}$$

The next step to the Riemann-Hilbert transform is to define the wavefunction $\chi(\lambda)$ as in equation (6) and its analytic conditions χ_+ and χ_-, related through (7). Following the same line of reasoning as in [2] it can be proved that the Riemann-Hilbert transform keeps the linear equation unaltered. So we can now proceed to set up the conditions to be satisfied by the numerical matrix $\gamma(\lambda)$ or its infinitesimal form $\theta(\lambda)$ and the wavefunction $X(\lambda)$.

The conditions read

$$\begin{aligned} &\chi(\lambda) \in \mathrm{SL}(3, R) \\ &Q\gamma(\lambda)Q^{-1} = \gamma(q\lambda) \qquad \text{where } t = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &t\gamma(\lambda)t^{-1} = -\gamma^{t\gamma}(-\lambda) \end{aligned} \tag{20}$$

and the corresponding reduction is known as D_3 reduction.

Expanding the infinitesimal form of $\gamma(\lambda)$, i.e. $\theta(\lambda)$ as $\theta(\lambda)=\Sigma_{n=-\infty}^{\infty}\lambda^n\theta_n$

$$\begin{aligned} Q^{-1}\theta_n Q &= q^n\theta_n \\ t^{-1}\theta_n t &= \theta_n^{\mathrm{tr}}. \end{aligned} \tag{21}$$

The first condition of (21) yields

$$\begin{aligned} Q^{-1}\theta_{3K}Q &= \theta_{3K} \\ Q^{-1}\theta_{3K+1}Q &= q\theta_{3K+1} \\ Q^{-1}\theta_{3K+2}Q &= \bar{q}\theta_{3K+2} \end{aligned} \tag{22}$$

where K is an integer. With the form of Q given before we can actually solve the set (22) and furthermore the imposition of the second condition yields the solutions in the following form

$$\begin{aligned} \theta_{3K} &= pE_{+1}+r(E_{-2}+E_{+3}) \\ \theta_{3K+1} &= dE_{-1}+h(E_{+2}+E_{-3}) \\ \theta_{3K+2} &= \alpha\lambda_3 \end{aligned} \tag{23}$$

where p, r, d, h, α are arbitrary constants and $E_{\pm1}=\lambda_1\pm\mathrm{i}\lambda_2$; $E_{\pm2}=\lambda_4\pm\mathrm{i}\lambda_5$; $E_{\pm3}=\lambda_6\pm\mathrm{i}\lambda_7$ and λ_s are the Gell-Mann SU(3) matrices. So the final form of the Kac-Moody algebra can be presented in the form; $\theta(\lambda)=\Sigma\,\theta_{3K}\lambda^{3K}+\Sigma\,\theta_{3K+1}\lambda^{3K+1}+\Sigma\,\theta_{3K+2}\lambda^{3K+2}$ so that:

$$T_a^m=\begin{cases}\lambda^m(E_{+1}, E_{-2}, E_{+3}) & m=0 \quad \mathrm{mod}\ 3\\ \lambda^m(E_{-1}, E_{+2}, E_{-3}) & m=1 \quad \mathrm{mod}\ 3\\ \lambda^m(\alpha\lambda_3) & m=2 \quad \mathrm{mod}\ 3.\end{cases}$$

4. Supersymmetric extensions

Our next motivation is to obtain the Kac-Moody algebra for the hidden symmetries in the supersymmetric version of the Dodd-Bullough equation. We have chosen this particular case as it is sufficiently general to unfold the intricacies involved. The supersymmetric extensions of the generalised Toda system have been discussed in detail by Olshanetsky [6]. Here it has been observed that if one has the following Lax pair

$$D_1\psi = U\psi \qquad D_2\psi = V\psi \tag{24}$$

where

$$\begin{aligned} D_1 &= -\partial_{\theta_2}+\mathrm{i}\theta_2\partial_t & D_2 &= \partial_{\theta_1}+\mathrm{i}\theta_1\partial_x \\ U &= U_0+\lambda U_1 & V &= \lambda^{-1}V_1 \end{aligned} \tag{25}$$

are superderivatives and θ_1, θ_2 are the anticommuting coordinates. ψ is the super wavefunction, with U and V also matrices depending on the supercoordinates, fields and the eigenvalue λ.

The compatibility now reads

$$D_2U+D_1V=\{U, V\} \tag{26}$$

where $\{A, B\}$ denotes anticommutator of two arbitrary matrices. It is easily seen that (25) leads to

$$D_1 V_1 = \{U_0, V_1\} \qquad D_2 U_0 = \{V_1, U_1\}. \tag{27}$$

Now to reproduce a particular nonlinear equation from (27) one takes recourse to a particular super Lie algebra in which we assume the U and V to belong. For the case under consideration the Lie algebra in Kac's classification is $A^{(4)}$ (0, 2) which have the generators E_i^+, E_i^-, H_i following the following set of commutation rules:

$$\begin{aligned} &\{E_1^+, E_1^-\} = H_1 \\ &[E_i^+, E_j^-] = \delta_{ij} H_j, \qquad i, j \neq 1 \\ &[H_i, E_j^\pm] = a_{ij} E_j^\pm. \end{aligned} \tag{28}$$

It is then obvious that we can write

$$\begin{aligned} U_0 &= \sum A_j(x, t) H_j + \sum A_j'(x, t) E_j^+ + \sum A_j''(x, t) E_j^- \\ V_1 &= \sum C_j(x, t) E_j^- + \sum C_j'(x, t) H_j + \sum C_j''(x, t) E_j^+ \\ U_1 &= \sum B_j(x, t) E_j^+ - \sum B_j'(x, t) H_j + \sum B_j''(x, t) E_j. \end{aligned} \tag{29}$$

Imposing the Z_n reduction on the matrices U and V with the help of the matrix Q defined through

$$Q(E_j^+) = q^- E_j^+ \qquad Q(H_j) = H_j \qquad Q(E_j^-) = qE_j^- \tag{30}$$

we observe that V_0, U_1 and V_1 do possess the following decomposition

$$U_0 = \sum A_j(x, t) H_j \qquad U_1 = \sum B_j(x, t) E_j^+ \qquad V_1 = \sum C_j(x, t) E_j^-. \tag{31}$$

Now using the compatibility equations (26) and the superalgebra (28) one can reproduce the following nonlinear equation known as the supersymmetric Dodd-Bullough equation

$$\theta_{xt} = -\mathrm{e}^{\theta}(\mathrm{e}^{\theta} + \phi_1\phi_2) + \mathrm{e}^{\theta} \qquad \phi_{1x} = -2\mathrm{e}^{\theta}\phi_2 \qquad \phi_{2x} = -2\mathrm{e}^{\theta}\phi_1 \tag{32}$$

where θ and ϕ_1, ϕ_2 are, respectively, the bosonic and fermionic part of the superfield Φ. To explore the structure of the Kac-Moody algebra we again impose the restriction of the reduction condition on $\gamma(\lambda)$ which now belongs to a graded Lie algebra. The analogue of condition (20) reads

$$\chi(\lambda) \in A^{(4)}(0, 2) \qquad Q\chi(\lambda)Q^{-1} = q\gamma(\lambda) \qquad q = \mathrm{e}^{\mathrm{i}\pi}. \tag{33}$$

Taking recourse to the infinitesimal expansion of $\gamma(\lambda)$ and using (33) we get

$$\gamma(\lambda) = 1 + \sum \Omega_n^j E_j^+ \lambda^n + \sum \Delta_n^j E_j^- \lambda^n + \sum K_n^j H_j \lambda^n \tag{34}$$

along with

$$Q\Omega_n^j Q^{-1} = \Omega_n^j q^{n+1} \qquad Q\Delta_n^j Q^{-1} = \Delta_n^j q^{n-1} \qquad QK_n^j Q^{-1} = K_n^j. \tag{35}$$

Again the grading is obtained for different odd and even values of n. We can actually solve these sets of equations; for example for $n=1$

$$\Omega_1=\begin{pmatrix} a & 0 & c & 0 \\ 0 & f & 0 & h \\ q & 0 & K & 0 \\ 0 & n & 0 & p \end{pmatrix} \tag{36}$$

for $n=2$

$$\Omega_2=\begin{pmatrix} 0 & b & 0 & d \\ e & 0 & g & 0 \\ 0 & j & 0 & e \\ n & 0 & K & 0 \end{pmatrix}. \tag{37}$$

Similarly for other cases, a, c, f, g, etc are arbitrary constants. It is quite easy to observe that Ω_1, Ω_2, Δ_1, Δ_2 etc again belong to a particular subalgebra of the $A^{(4)}$ (0, 2) [8].

In order to obtain an explicit representation of Ω_1 and Ω_2 we note that the super Lie algebra SU(3|1) or $A^{(4)}$ (0, 2) has got the following generators in the explicit form [8];

$$\beta_{i(i=1,\ldots,8)}=\left(\begin{array}{ccc|c} & & & 0 \\ & \lambda_i & & 0 \\ & & & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right) \tag{38}$$

where λ_i are the usual SU(3)-Gell-Mann matrices

$$\beta_n=\begin{pmatrix} -\frac{1}{2} & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & -\frac{3}{2} \end{pmatrix} \tag{39}$$

and $(\beta_{\alpha\alpha})_{ab}=\delta^4_a\delta^\alpha_b$ where a, b indicates the rows and columns ($a, b=1,\ldots,4$) and $\alpha=1,2,3$, no summation is implied over α.

Finally,

$$\beta^{\alpha\alpha}=(\beta_{\alpha\alpha})^{\mathrm{T}}. \tag{40}$$

So in total we have eight β_i, one β_n, three $\beta_{\alpha\alpha}$ and three $\beta^{\alpha\alpha}$ in a total of fifteen generators. Now it is very easy to observe that

$$\Omega_1=\tfrac{1}{2}c(\beta_5+\mathrm{i}\beta_4)+\tfrac{1}{2}q(\beta_5-\mathrm{i}\beta_4)+n\beta_{22}+q\beta^{22}+(3\alpha)^{1/2}\beta_8+2\delta\beta_n+\gamma\beta_3. \tag{41}$$

So in this case the subalgebra consists of the generators; T^m_a ($m=1$, $a=1,\ldots,7$)

$$\begin{array}{llll} T^1_1=\beta_{22} & T^1_2=\beta^{22} & T^1_3=\beta_8 & T^1_4=\beta_n \\ T^1_5=\beta_3 & T^1_6=\frac{1}{2}(\beta_5+\mathrm{i}\beta_4) & T^1_7=\frac{1}{2}(\beta_5-\mathrm{i}\beta_4) & \end{array} \tag{42}$$

where α, δ, γ are connected to a, f, K, p by some simple linear relations

$$\gamma=\tfrac{1}{2}(a-f);\qquad \delta=-\tfrac{1}{3},\qquad \alpha=-\tfrac{1}{2}(K-\tfrac{1}{3}p).$$

In the other case;

$$\Omega_2 = d\beta_{11} + n\beta^{11} + \tfrac{1}{2}b(\beta_1 + \mathrm{i}\beta_2) + \tfrac{1}{2}e(\beta_1 - \mathrm{i}\beta_2) + \tfrac{1}{2}g(\beta_6 + \mathrm{i}\beta_7) + \tfrac{1}{2}j(\beta_6 - \mathrm{i}\beta_7) + e\beta_{33} + K\beta^{33}. \tag{43}$$

So this time the subalgebra consists of the following generators;

$$\begin{array}{lll} (T_a^m, m=2, a=1,\ldots,8) & & \\ T_1^2 = \beta_{11} & T_2^2 = \beta^{11} & T_3^2 = \beta_{33} \\ T_4^2 = \beta^{33} & T_5^2 = \tfrac{1}{2}(\beta_1 + \mathrm{i}\beta_2) & T_6^2 = \tfrac{1}{2}(\beta_1 - \mathrm{i}\beta_2) \\ T_7^2 = \tfrac{1}{2}(\beta_6 + \mathrm{i}\beta_7) & T_8^2 = \tfrac{1}{2}(\beta_6 - \mathrm{i}\beta_7) & \end{array} \tag{44}$$

So even in the supersymmetric nonlinear case it is possible to have an explicit realisation of the graded Kac-Moody generators.

5. Conclusion

In this paper we have given a detailed exposition of a concrete method to derive hidden symmetries from linearisation systems of various integrable models. Since here we were able to cover a wider range of integrable systems with more completed algebraic structure and with supersymmetry one is led to believe that all two-dimensional integrable models could have an infinite parameter hidden symmetry of the Kac-Moody type. One should think that parts of the method can be carried over to higher dimensional integrable models. Take for example the three-dimensional KdV equation (the Kadomstev-Petviashvili equation). Here we also find a reduction system [7] and our method here seems also to work with such a reduction system. However, since such a reduction system has no explicit λ dependence which can only be recovered in the asymptotic limit $\chi \to \infty$, it is necessary to prove the group property on the scattering data rather than on the wavefunction.

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